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Random Perturbations of Periodically Driven Nonlinear Oscillators

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Abstract

This paper develops a unified approach to study the dynamics of nonlinear oscillators excited by both periodic and random perturbations. This study is motivated by problems that range from nonlinear energy harvesting to ship capsizing in random seas. The near resonant dynamics of such systems, in the presence of weak noise, is not well understood. Nonlinear systems driven by sufficiently strong periodic parametric excitation often display a range of phenomena from period doubling to chaos. In the presence of weak noise there are transitions between the domains of attraction of the stable periodic orbits. The effects of noisy perturbations on the passage of trajectories through the resonance zones is studied in depth using the large deviation theory.

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1. Introduction

The recent surge of research articles in energy harvesting focuses on the "cantilever beam" type devices which are used to convert small amplitude mechanical vibration from a specific application into an electrical energy source that could be used for electronic devices with low power requirements (see references in [1]). Prototypical beam type nonlinear energy harvesting models contain double well potentials, external or parametric periodic forcing terms, damping and ambient broadband additive noise terms. For example [2,3] considers

$$\ddot{q}_t + \delta \dot{q}_t - \mu(1 - \eta \cos(\nu t))q_t + \gamma q_t^3 = \sigma\, \xi(t) + \alpha \cos(\nu t), \tag{1}$$

where $q_t \in \mathbb{R}$ represents the non-dimensional generalized coordinate, δ is the damping, μ is the measure of the compressive load acting on the beam, γ is the nonlinear modal stiffness coefficient, and η and ν are the magnitude and frequency of the force modulation respectively. This oscillatory force causes the beam to compress and relax in an oscillatory manner. ξ represents mean zero, stationary, Gaussian white noise process.

It is well known that container ships tend to experience parametric roll motion in random seas. The rolling motion of a ship in head or following waves can be represented by the differential equation for the roll angle that includes non-linear wave drag force and non-linear restoring moment and stochastic forcing. It is worth noting that the asymptotic

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technique presented in this paper by combining homogenization and large deviations, can be also used to determine the mean time for a specific vessel to capsize or to reach a critical roll angle. The details of the theory presented here, and it's application to nonlinear energy harvesting and ship capsizing in random seas are presented in[4].

In this paper we study the dynamics of the oscillator (1) when the noise and damping are small and the periodic force is in resonance with the oscillator. For this purpose we introduce smallness parameter $\varepsilon \ll 1$ and consider

$$\ddot{q}_t - \mu q_t + \gamma q_t^3 = \varepsilon(\alpha \cos(\nu t) + \mu\eta \cos(\nu t) q_t - \delta \dot{q}_t) + \varepsilon^{\kappa} \sigma \xi(t). \tag{2}$$

The weakly nonlinear deterministic *Duffing–Mathieu* equation (2) with $\gamma \sim O(\varepsilon)$ and $\sigma = 0$, has been studied extensively in the literature (see for example,[5] and[6]). On the other hand, in the absence of periodic perturbations ($\alpha = 0, \eta = 0$), (2) represents a special case of the noisy *Duffing-van der Pol* equation which has been studied by[7] and[8], to name a few. Reference[9] developed a unified approach, for the weakly nonlinear (ε order cubic nonlinearity) noisy *Duffing–van der Pol–Mathieu* equation (2), by a clever treatment in a neighborhood of the separatrix where the unperturbed orbits have arbitrarily long periods. Here, by appropriate scaling of the nonlinear term in (2), the solution $(q^\varepsilon, \dot{q}^\varepsilon)$ over any finite interval converges in probability, as $\varepsilon \searrow 0$, to the solution of an averaged equation which has a conservation law. The averaged equation had certain nontrivial (yet generic) types of fixed points. The evolution of the first integral (conservation law) was examined on a rescaled time interval.

The unperturbed system corresponding to (2)

$$\ddot{q}_t - \mu q_t + \gamma q_t^3 = 0, \tag{3}$$

has three fixed points: the fixed point $q = 0$ is a saddle and the other two fixed points $q = \pm\sqrt{\mu/\gamma}$ (corresponding to the bottom of the wells in the double-well potential) are centers. The equation (2) can be studied as a perturbation of the Hamiltonian system

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{4}$$

with the Hamiltonian with a double-well potential U:

$$H(q, p) = \frac{1}{2}p^2 + U(q), \quad U(q) = -\frac{\mu}{2}q^2 + \frac{\gamma}{4}q^4. \tag{5}$$

Let (I, φ) be action angle variables corresponding to the unperturbed system (4) and assume that

$$I = I(q_1, q_2), \quad \varphi = \varphi(q_1, q_2), \qquad \text{along with} \qquad q_1 = q_1(I, \varphi), \quad q_2 = q_2(I, \varphi)$$

can be written where $q_2 = p$. Then, the system (2) with $\varepsilon = 0$ can be written as

$$\dot{I} = 0, \qquad \dot{\varphi} = \Omega(I). \tag{6}$$

Let θ be the angle variable corresponding to $\dot{\theta} = \nu$ that represents the periodic forcing term in (2). In the perturbed system (2) with $\varepsilon \neq 0$, the frequency $\Omega(I)$ changes with time and if the frequencies $\Omega(I)$ and ν are non-commensurable, then the (φ, θ) orbits densely fill the state-space and the motion is called *quasi-periodic*. Resonance occurs when the frequencies ν and $\Omega(I)$ are commensurable or nearly commensurable and in this case orbits do not densely fill the state-space. Since Ω depends on the action I, the resonance will depend on certain values of the action. Some trajectories get captured into a resonance zone and others 'pass-through'.

Typically, studies which treat nonlinear oscillators as perturbation of Hamiltonian systems involve some kind of averaging principle. Issues that arise in obtaining averaging principle in presence of resonances are discussed, for example, in[10] and[11]. In studying the dynamics close to a resonance zone, partial averaging is employed as discussed in[11]. For example let I_r be such that $n\nu = m\Omega(I_r)$ where n, m are integers. Then in a region of the phase-space where I is close to I_r, introduce a new variable $\psi = \varphi - \frac{n}{m}\theta$. The dynamics in this region can be described using (I, ψ) which are slow-variables while averaging out the fast variable θ.

Periodically driven nonlinear systems *with noise* are considered in[12,13]. Reference[13] considers $\kappa = 1/2$ (in this paper we consider $\kappa > 1$, i.e., strength of the noise in[13] is stronger than that assumed here) and assumes that the noise in (I, φ) variables is uniformly non-degenerate and obtains an averaging principle to the effect that the resonances could be totally ignored. However, the system (2) that is being considered here, does not obey that hypothesis (see the paragraph immediately following theorem 2.1 of[13] showing the restrictive nature of that hypothesis).

In light of the above discussion,[4] studies the effect of weak noise on the escape from a resonance zone. In the absence of the noise, the trajectories which get trapped in a resonance zone never leave it — however, the noise facilitates the escape. Two-regimes $\kappa > 1$ and $\kappa = 1$ in (2) are considered in[4]. For $\kappa = 1$,[4] found the large deviation rate functional for escape from a resonance zone.[4] showed that trajectories of the system trickle down close to the bottom of the potential wells.

Recall that the point $q = \sqrt{\mu/\gamma}$ corresponds to the bottom of the potential well for the unperturbed system (2). Linearizing (2) about $q = \sqrt{\mu/\gamma}$, i.e., setting $x = q - \sqrt{\mu/\gamma}$ and retaining terms linear in x, we get $\ddot{x} + 2\mu x = 0$. This shows that, close to the bottom of the potential well the unperturbed system behaves approximately like a harmonic oscillator with frequency $\sqrt{2\mu}$. However, a close scrutiny shows the noise induced transitions in the limit of small noise intensity, is very complicated. Unraveling these transitions between stable limit cycles (quasi-periodic) and the study of exit times from the domains of attractions are the focus of this theoretical study.

2. Resonance at the bottom of the potential well

In this section we study the perturbed system (2) when the forcing frequency ν is close to $2\sqrt{2\mu}$, i.e., we assume $\nu = 2\sqrt{2\mu}(1 + \varepsilon\lambda)$, where λ is a detuning parameter. Such a situation is also discussed in[14] in an attempt to explain phase-flip of electrons in external fields.

We shift the origin of the coordinate system to the bottom of one of the potential wells using the transformation $x^\varepsilon_{1,t} = q^\varepsilon_t - \sqrt{\mu/\gamma}$ and $x^\varepsilon_{2,t} = \dot{q}^\varepsilon_t$ in equation (2). Then (2) in state-space form is

$$\begin{cases} dx^\varepsilon_{1,t} &= x^\varepsilon_{2,t}dt \\ dx^\varepsilon_{2,t} &= -(2\mu x^\varepsilon_{1,t} + 3\gamma\sqrt{\mu/\gamma}(x^\varepsilon_{1,t})^2 + \gamma(x^\varepsilon_{1,t})^3)dt + \varepsilon\sqrt{\mu/\gamma}\eta\mu\cos(\nu t)dt \\ & \qquad +\varepsilon\left(\eta\mu\cos(\nu t)x^\varepsilon_{1,t} + \alpha\cos(\nu t) - \delta x^\varepsilon_{2,t}\right)dt + \varepsilon^\kappa\sigma dW_t, \end{cases} \tag{7}$$

where W_t is a Wiener process. In this paper we set $\alpha = 0$ for convenience. The forcing $\varepsilon\sqrt{\mu/\gamma}\eta\mu\cos(\nu t)$ induces a periodic motion (approximate) of $O(\varepsilon)$ amplitude. The significant length scale in the system turns out to be $O(\sqrt{\varepsilon})$. Further, the system (7) could be simplified by performing a near identity transformation which eliminates the quadratic nonlinearities in (7). This motivates the following sequence of transformations on (7):

$$\begin{cases} v^\varepsilon_{1,t} &= \frac{1}{\sqrt{\varepsilon}}\left(x^\varepsilon_{1,t} + \varepsilon\frac{\eta\mu\sqrt{\mu/\gamma}}{\nu^2-2\mu}\cos(\nu t)\right), \\ v^\varepsilon_{2,t} &= \frac{1}{\sqrt{\varepsilon}}\frac{1}{\sqrt{2\mu}}\left(x^\varepsilon_{2,t} - \varepsilon\frac{\eta\mu\sqrt{\mu/\gamma}}{\nu^2-2\mu}\nu\sin(\nu t)\right), \end{cases} \qquad \text{and} \qquad \begin{pmatrix} v^\varepsilon_{1,t} \\ v^\varepsilon_{1,t} \end{pmatrix} = \begin{pmatrix} y^\varepsilon_{1,t} \\ y^\varepsilon_{2,t} \end{pmatrix} - \sqrt{\varepsilon}\sqrt{\frac{\gamma}{4\mu}}\begin{pmatrix} (y^\varepsilon_{1,t})^2 + 2(y^\varepsilon_{2,t})^2 \\ -2y^\varepsilon_{1,t}y^\varepsilon_{2,t} \end{pmatrix}.$$

We find that the dominant dynamics of y is rotation with frequency close to $\frac{1}{2}\nu$. Also, significant changes in the amplitude occurs on times of order $1/\varepsilon$. So, we make one additional transformation to remove the rotaion and scale time appropriately:

$$\begin{pmatrix} z^\varepsilon_{1,t} \\ z^\varepsilon_{1,t} \end{pmatrix} = e^{-tB/\varepsilon}\begin{pmatrix} y^\varepsilon_{1,t/\varepsilon} \\ y^\varepsilon_{2,t/\varepsilon} \end{pmatrix}, \qquad \text{where} \qquad B = \frac{1}{2}\nu\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Applying the above transformations in (7) yields

$$dz^\varepsilon_t = e^{-tB/\varepsilon}\left\{\begin{pmatrix} 0 & 0 \\ (\eta\sqrt{2\mu})\cos(\nu t/\varepsilon) & -\delta \end{pmatrix} - \frac{3\gamma}{4\mu}(z_1^2 + z_2^2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \lambda B\right\} e^{tB/\varepsilon}z^\varepsilon_t dt + \varepsilon^{\kappa-1}\frac{\sigma}{\sqrt{2\mu}}e^{-tB/\varepsilon}\begin{pmatrix} 0 \\ 1 \end{pmatrix}dW_t \; + \; h.o.t \tag{8}$$

The higher order terms are not significant for dynamics of z^ε on times $[0, T]$. The fast oscillating coefficients in the above equation can be averaged out. Define the averaged drift coefficient by

$$\mathscr{B}(z) = \left(-\frac{3\gamma}{4\mu}(z_1^2 + z_2^2) - \frac{\nu}{2}\lambda\right)\begin{pmatrix} z_2 \\ -z_1 \end{pmatrix} - \frac{\delta}{2}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \frac{\eta\sqrt{2\mu}}{4}\begin{pmatrix} z_2 \\ z_1 \end{pmatrix}.$$

Then it can be shown that

Theorem 2.1. *The distribution of z^ε for $t \in [0, T]$ converges as $\varepsilon \to 0$ to the distribution of z given by the deterministic system*

$$\dot{z} = \mathscr{B}(z). \tag{9}$$

Lets study the deterministic system (9). One obvious fixed point of (9) is $(0, 0)$. Others are given by solving the system of equations

$$\sqrt{(z_1^2 + z_2^2)} = \frac{\sqrt{4\mu}}{\sqrt{3\gamma}} \sqrt{-(\nu\lambda/2) \pm \sqrt{(\eta\sqrt{2\mu}/4)^2 - (\delta/2)^2}} \quad =: \mathbf{R}_\pm, \tag{10}$$

$$\frac{2z_1 z_2}{z_1^2 + z_2^2} = \frac{\delta/2}{\eta\sqrt{2\mu}/4}. \tag{11}$$

Note that for $\sqrt{(z_1^2 + z_2^2)}$ to be real, we need $\frac{1}{4}\eta\sqrt{2\mu} > \frac{1}{2}\delta$. If

$$\frac{1}{4}\eta\sqrt{2\mu} > \frac{1}{2}\delta \qquad \text{and} \qquad -(\nu\lambda/2) > \sqrt{(\eta\sqrt{2\mu}/4)^2 - (\delta/2)^2}, \tag{12}$$

then two values are possible for $\sqrt{(z_1^2 + z_2^2)}$. Also note that if (z_1, z_2) is fixed point then so is $(-z_1, -z_2)$. So, in total there are four nontrivial fixed points. The points with $\sqrt{(z_1^2 + z_2^2)} = \mathbf{R}_-$ are saddles for (9) and the points with $\sqrt{(z_1^2 + z_2^2)} = \mathbf{R}_+$ are sinks for (9). This means that for the system obtained by setting $\sigma = 0$ in (2), the following solutions are possible (when higher order terms are ignored):

$$q_t^\varepsilon = \sqrt{\mu/\gamma} + \mathbf{0} - \varepsilon\frac{\eta\mu\sqrt{\mu/\gamma}}{\nu^2 - 2\mu}\cos(\nu t), \tag{13}$$

$$q_t^\varepsilon = \sqrt{\mu/\gamma} + \sqrt{\varepsilon}\mathbf{R}_+\cos(\nu t/2 + \theta^+) - \varepsilon\frac{\eta\mu\sqrt{\mu/\gamma}}{\nu^2 - 2\mu}\cos(\nu t), \tag{14}$$

$$q_t^\varepsilon = \sqrt{\mu/\gamma} + \sqrt{\varepsilon}\mathbf{R}_+\cos(\nu t/2 + \theta^+ + \pi) - \varepsilon\frac{\eta\mu\sqrt{\mu/\gamma}}{\nu^2 - 2\mu}\cos(\nu t), \tag{15}$$

$$q_t^\varepsilon = \sqrt{\mu/\gamma} + \sqrt{\varepsilon}\mathbf{R}_-\cos(\nu t/2 + \theta^-) - \varepsilon\frac{\eta\mu\sqrt{\mu/\gamma}}{\nu^2 - 2\mu}\cos(\nu t), \tag{16}$$

$$q_t^\varepsilon = \sqrt{\mu/\gamma} + \sqrt{\varepsilon}\mathbf{R}_-\cos(\nu t/2 + \theta^- + \pi) - \varepsilon\frac{\eta\mu\sqrt{\mu/\gamma}}{\nu^2 - 2\mu}\cos(\nu t). \tag{17}$$

The solutions in (16) and (17) are unstable. Others are stable. Let the fixed points of (9) corresponding to (13)-(17) be denoted respectively by z^0, z^{+0}, $z^{+\pi}$, z^{-0}, $z^{-\pi}$. Then z^0, $z^{+0}, z^{+\pi}$ are stable and $z^{-0}, z^{-\pi}$ are saddles. In presence of noise, i.e., $\sigma \neq 0$, transitions occur between the domains of attraction of z^0, z^{+0} and $z^{+\pi}$ (equivalently between the solutions (13)-(15)).

Let K_0 be the domain of attraction (see figure 1) of the stable trivial equilibrium $(0, 0)$. Let K_1, K_2 be the domains of attraction of the fixed points z^{+0} and $z^{+\pi}$.

By theorem 2.1, the trajectories of z^ε that start in the domain of attraction of a stable fixed point, with probability close to 1, reach a neighborhood of the fixed point while staying close to the corresponding trajectory of the deterministic system (9). However, there is a very small probability that noise leads to transition of the trajectories from one domain of attraction to other. This article studies the probability of such rare events happening on times $0 \leq t \leq T$. The natural framework for such a study is large deviations.

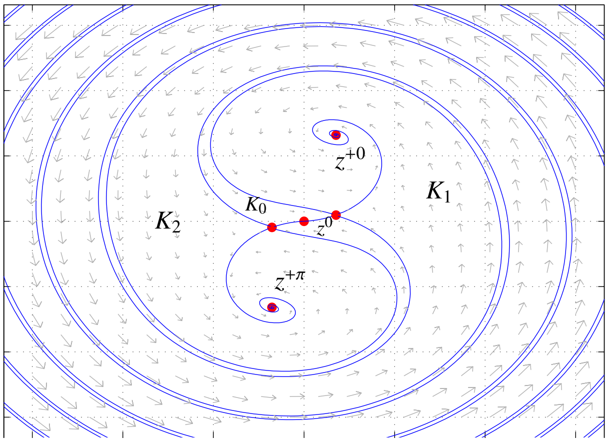

Fig. 1. Typical phase portrait for (9) when the conditions (12) are satisfied. The blue lines are stable and unstable manifolds of the saddle points z^{-0} and $z^{-\pi}$. The domain of attraction for z^0, z^{+0}, $z^{+\pi}$ are separted by the blue lines. Figure generated using the software at [15].

2.1. Large Deviations

The large deviation theory is concerned with asymptotic estimates of probability of rare events associated with stochastic processes. Let x^ε be a family of processes depending on a small parameter ε. Let, A be an event associated with the process x^ε, for example escape from the domains of attractions (K_0, K_1 and K_2). Let $\mathbb{P}(x^\varepsilon \in A)$ be the probability of the event A. Roughly speaking, if there exists a functional I on the space of trajectories of x^ε such that

$$-\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(x^\varepsilon \in A) = \inf_{\varphi\in A} I(\varphi), \tag{18}$$

(the infimum is taken over the trajectories φ which satisfy A) then the process x^ε is said to satisfy a large deviation principle (LDP) with rate function I. It means that, as $\varepsilon \to 0$ the probability of A happening is of the order of $e^{-\frac{1}{\varepsilon}\inf_{\varphi\in A} I(\varphi)}$ which goes to zero as $\varepsilon \to 0$. Roughly speaking, that path φ satisfying A which minimizes I is the most likely way that A happens – path of maximum likelihood (PML) reduces to a deterministic optimization problem.

Using averaging techniques and standard results from large deviations, the following theorem can be proved:

Theorem 2.2. *Consider the process z^ε defined by (8). The rate functional on the path space defined by*

$$-\lim_{\varepsilon\to 0} \varepsilon^{2(\kappa-1)} \log \mathbb{P}(z^\varepsilon \in A) = \inf_{\varphi\in A} S_{T_1T_2}(\varphi), \tag{19}$$

for A a Borel subset of $C([T_1, T_2], \mathbb{R}^2)$, equals

$$S_{T_1T_2}(\varphi) = \frac{1}{2}\int_{T_1}^{T_2} \frac{\|\dot\varphi_t - \mathscr{B}(\varphi_t)\|^2}{\sigma^2/(4\mu)} dt. \tag{20}$$

for $\varphi \in C([T_1, T_2], \mathbb{R}^2)$ absolutely continuous.

Before giving a sketch of the proof, lets see how this result can be useful. Define

$$\mathscr{V}(t, x, y) := \inf\{S_{0t}(\varphi) \ : \ \varphi \in C([0, t], \mathbb{R}^2),\ \varphi(0) = x,\ \varphi(t) = y\}. \tag{21}$$

The function $\mathscr{V}(t, x, y)$ satisfies a Hamilton-Jacobi PDE and may be solved numerically.

Suppose we want to consider the probability of escape from the domain of attraction K_i of a fixed point z_* in time T. Let x be the starting point in K_i and $\mathfrak{e}^\varepsilon$ be the time of escape. Applying theorem 4.1.2 and the remarks following it in [18] we have[1] that $\mathbb{P}_x[\mathfrak{e}^\varepsilon \leq t]$ satisfies

$$\lim_{\varepsilon\to 0} \varepsilon^{2(\kappa-1)} \log \mathbb{P}_x[\mathfrak{e}^\varepsilon \leq t] \;=\; -\min_{\substack{0\leq s\leq t \\ y\notin K_i}} \mathscr{V}(s,x,y).$$

The next two results (theorems 4.2.1 and 4.4.1 of [18]) deal with probability distribution of the location of exit on the boundary and mean exit times from the domain of attraction. These are derived under the assumption that the vector field on the boundary is pointing into the domain of attraction. This is not satisfied in our case. For the problem considered in this paper the boundary is a characteristic, i.e., the (averaged) vector field on the boundary points along the boundary. Nevertheless we would apply the result.

Define the *quasipotential*

$$\mathscr{V}(z_*, x) := \inf\{S_{T_1T_2}(\varphi) \;:\; \varphi \in C([T_1, T_2], \mathbb{R}^2),\; \varphi(T_1) = z_*,\; \varphi(T_2) = x,\; T_1 \leq T_2\}, \tag{22}$$

where z_* is the fixed point of the domain of attraction K_i with boundary ∂K_i. Let y_* be the minimizer of $\min_{y\in\partial K_i} \mathscr{V}(z_*, y)$. Then, theorem 4.2.1 in [18] shows that, as $\varepsilon \to 0$, the probability that exit occurs in a neighborhood close to y_* goes to 1. Theorem 4.4.1 of [18] shows that[2] the mean exit time satisfies

$$\lim_{\varepsilon\to 0} \varepsilon^{2(\kappa-1)} \log \mathbb{E}_x \mathfrak{e}^\varepsilon \;=\; \min_{y\in\partial K_i} \mathscr{V}(z_*, y). \tag{23}$$

The goal of the next section is to study the dependence of the quasipotential on the parameters of the system: damping (δ), detuning (λ) and strength of nonlinearity (γ) while fixing the values of μ and strength of periodic excitation (η).

Before moving to the next section, we give a brief sketch of the proof of theorem 2.2 the details of which can be found in [4].

Roughly speaking, if a family X^ε of $\mathbb{R}$-valued random variables indexed by $\varepsilon \in \mathbb{R}$ has a LDP with rate function $\mathcal{V}$, then, the probability density function f would roughly be $f(x) \sim e^{-\mathcal{V}(x)/\varepsilon}$. Calculating the log of (scaled) moment generating function $g^\varepsilon(p) := \varepsilon \log \mathbb{E}[e^{pX^\varepsilon/\varepsilon}]$ we get that $g^\varepsilon(p) \sim \varepsilon \log \int e^{px/\varepsilon} e^{-\mathcal{V}(x)/\varepsilon} dx$. It is clear that as $\varepsilon \to 0$, $g^\varepsilon(p)$ tends to $\sup_{x\in\mathbb{R}}(px - \mathcal{V}(x))$. So, define $g(p) := \lim_{\varepsilon\to 0} g^\varepsilon(p)$. Then $g(p) = \sup_{x\in\mathbb{R}}(px - \mathcal{V}(x))$. Inverting, we get that $\mathcal{V}(x) = \sup_{p\in\mathbb{R}}(px - g(p))$. So, one way to find the LDP rate function is to calculate the moment generating function and obtain the supremum above. We apply this technique to calculate the rate function for the two-dimensional variable $z^\varepsilon(t_2)$ when the process starts at time t_1 at location $x \in \mathbb{R}^2$. Define

$$g^\varepsilon_{x,t_1,t_2}(p) = \varepsilon^{2(\kappa-1)} \log \mathbb{E}_{(t_1,x)}\left[\exp\left(\frac{p_1 z_1^\varepsilon(t_2) + p_2 z_2^\varepsilon(t_2)}{\varepsilon^{2(\kappa-1)}}\right)\right].$$

Then, $g_{x,t_1,t_2}(p) := \lim_{\varepsilon\to 0} g^\varepsilon_{x,t_1,t_2}(p)$ can be shown to be

$$g(p) = p^{tr}x + p^{tr}\int_{t_1}^{t_2} \mathscr{B}(\psi_s)ds + \frac{1}{2}\frac{\sigma^2}{4\mu}\|p\|_2^2(t_2 - t_1),$$

where ψ is simulated according to

$$\dot{\psi} = \mathscr{B}(\psi) + \frac{\sigma^2}{4\mu}p, \qquad \psi_{t_1} = x.$$

Then, $I_{x,t_1,t_2}(z) := \sup_{p\in\mathbb{R}^2}(p^{tr}z - g_{x,t_1,t_2}(p))$ can be computed. Finally, the rate function $S_{T_1T_2}(\varphi)$ in theorem 2.2 can be obtained by partitioning the path into small intervals

$$S_{T_1T_2}(\varphi) = \lim_{n\to\infty} \sum_{j=0}^{n-1} I_{\varphi(t_j),t_j,t_{j+1}}(\varphi_{t_{j+1}}), \qquad t_j = T_1 + (j/n)(T_2 - T_1).$$

For details see [4].

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[1] This application is not rigorous. In [18] the vector field does not vary with ε. However, in the problem considered in this paper we are averaging a fast oscillating vector field to obtain $\mathscr{B}$.

[2] There is one caveat. While the mean exit times are of the order of $e^{\frac{1}{\varepsilon^{2(\kappa-1)}}}$, the averaging result of theorem 2.2 hold only on times of $O(1)$. Nevertheless, since the system is stable the result might still hold true.

2.2. Computation of the quasipotential

Recall the definition (22) of the quasipotential $\mathscr{V}$ and the action functional $S_{T_1T_2}$ defined in (20). The optimization problem in (22) can be written as follows:

$$\text{Minimize} \quad \frac{1}{2}\int_{T_1}^{T_2} \|u_s\|_2^2 ds \quad \text{subject to} \quad \dot{z}_t = \mathfrak{B}(z_t) + u_t \quad \text{with} \quad z_{T_1} = z_* \text{ and } \quad z_{T_2} = x.$$

Note that T_1 and T_2 are also free in the optimization, i.e., the minimum is over all possible T_1, T_2 with $T_1 \leq T_2$.

The usual method to solve this optimal control problem is as follows:

Define the Hamiltonian

$$H(z, p) := \sup_u \left(p^{tr}(\mathfrak{B}(z) + u) - \frac{1}{2}\|u\|_2^2 \right). \tag{24}$$

It is easy to see that the sup is obtained by taking $u = p$ and so

$$H(z, p) = p^{tr}\mathfrak{B}(z) + \frac{1}{2}\|p\|_2^2. \tag{25}$$

Then the trajectories for which $\frac{1}{2}\int_{T_1}^{T_2} \|u_s\|_2^2 ds$ has first variation zero satisfy the Euler-Lagrange equations

$$\dot{z} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial z}. \tag{26}$$

Furthermore, the fact that the time variables T_i are free, forces

$$H \equiv 0. \tag{27}$$

Of course we need to impose the boundary conditions

$$z(T_1) = z_*, \qquad z(T_2) = x \tag{28}$$

if we are interested in calculating the quasipotential $\mathscr{V}(z_*, x)$. Note that these are four boundary conditions—two for T_1 and two for T_2. And $\mathscr{V}(z_*, x)$ itself is obtained by integrating

$$\dot{V} = \frac{1}{2(\sigma^2/4\mu)}\|p\|^2 \quad \text{for } t \in [T_1, T_2], \quad V(T_1) = 0, \tag{29}$$

and setting $\mathscr{V}(z_*, x) = V(T_2)$ (note that u which gives sup in (24) is $u = p$).

Hence, the quasipotential could be obtained by solving (26) for the 4-dimensional system (z, p) with the four boundary conditions (28) while using (27) to determine the free parameters T_i and then using (29).

The above suggested method works except for the following issue. Recall that the z_* in the definition of quasipotential is a fixed point for $\dot{z} = \mathscr{B}(z)$. Hence $\mathscr{B}(z_*) = 0$. When $z = z_*$ (27) implies that $p = 0$. So $(z, p) = (z_*, 0)$ is a fixed point for the system of equations (26). So, the system started at (z_*, p) does not move from it.

To rectify this, [16] suggests the following as a numerical procedure to calculate the quasipotential. The optimization above does not occur for finite times T_i. Optimal trajectory takes infinite time to leave from (z_*, p). When it leaves, it leaves along the unstable manifold of (26) at $(z_*, p = 0)$. So, instead of starting at $(z_*, 0)$, start (26) at a point on the unstable manifold but very close to $(z_*, 0)$. The unstable manifold at $(z_*, 0)$ is tangential to the unstable eigenspace of the linearization of system (26) at $(z_*, 0)$. And this tangent can be easily found. Given $z^\dagger$ very close to z_*, there is a unique $p^\dagger$ so that $(z^\dagger, p^\dagger)$ belongs to the unstable eigenspace. So, we pick lot of $z^\dagger$ close to z_* and find corresponding $p^\dagger$s and simulate (26). Of all these simulations whichever trajectory passes through x is the desired trajectory.

The above is the numerical procedure that we use to study the dependence of the quasipotential on the system parameters in the next section. For the sake of completeness we write the system (26) explicitly, clearly showing its linear and nonlinear parts. Let $c = -\frac{3\gamma}{4\mu}\|z_*\|_2^2 - \frac{1}{2}\nu\lambda$. Then (26) can be written as

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} M & I_{2\times 2} \\ 0_{2\times 2} & N \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ p_1 \\ p_2 \end{pmatrix} + \frac{3\gamma}{4\mu} \begin{pmatrix} -(\|z\|_2^2 - \|z_*\|_2^2)z_2 \\ (\|z\|_2^2 - \|z_*\|_2^2)z_1 \\ 2z_1(p_1 z_2 - p_2 z_1) \\ 2z_2(p_1 z_2 - p_2 z_1) \end{pmatrix} \tag{30}$$

where $M = \begin{pmatrix} -\delta/2 & c + \eta\sqrt{2\mu}/4 \\ -c + \eta\sqrt{2\mu}/4 & -\delta/2 \end{pmatrix}$, and $N = \begin{pmatrix} \delta/2 & c - \eta\sqrt{2\mu}/4 \\ -c - \eta\sqrt{2\mu}/4 & \delta/2 \end{pmatrix}$. Let U be a matrix such that $(z = Up, p)$ is in the unstable eigenspace of the linearized system. Then we have $I + MU - UN = 0$. After solving this for U we can take a point in the unstable eigenspace as $(z, U^{-1}z)$. So, we choose z values near by z_* and then start (30) at $(z, U^{-1}z)$.

2.3. Dependence of the quasipotential on the system parameters

Recall from (23) that the mean exit times from a domain of attraction are determined by $\min_{y\in\partial K_i} \mathscr{V}(z_*, y)$. Recall also that the exit location is close to the above minimizer with probability approaching one. So we need to find $\min_{y\in\partial K_i} \mathscr{V}(z_*, y)$. The numerical procedure in the previous section can be used to find $\mathscr{V}(z_*, y)$ and the minmizer can be obtained by inspection. According to theorem 4.1 in [17] the minimizer would be the saddle point on the boundary[3]. Recall that z^{-0} and $z^{-\pi}$ are saddles. Define $V_{0,-} := \mathscr{V}(z^0, z^{-0})$ and $V_{+,-} := \mathscr{V}(z^{+0}, z^{-0})$. Then, mean exit time from the domain of attraction of the trivial fixed point z^0 is of the order of $e^{V_{0,-}/\varepsilon}$; and the mean exit time from the domain of attraction of one of the stable fixed points $z^{+0}, z^{+\pi}$ is of the order of $e^{V_{+,-}/\varepsilon}$. To get the order of mean time of transition from domain of attraction of z^{+0} to that of $z^{+\pi}$, a detailed analysis near the saddle point would be needed.

Explicit formula for quasipotential exists in one-dimensional systems and multi-dimensional systems whose vector field can be expressed as a gradient. For the non-gradient problem considered here, the explicit formulas for $V_{0,-}$ and $V_{+0,-0}$ could not be found. However, using numerical simulations some properties of them can be deduced as follows.

We fix (μ, η) and study how $V_{0,-}$ and $V_{+,-}$ vary with $(\delta, \lambda, \gamma)$. We focus only in the regime where there are 5 fixed points for z, i.e., portrait looks as in the figure 2. For this situation we need (according to (10)):

$$\delta \in [0, \sqrt{2\mu}\eta/2], \qquad \lambda < -\frac{1}{\sqrt{2\mu}}\sqrt{(\eta\sqrt{2\mu}/4)^2 - (\delta/2)^2}. \tag{31}$$

Since we fix (μ, η), we can simplify things by rescaling parameters: $\hat{\delta} = \delta/(\sqrt{2\mu}\eta/2)$, $\hat{\lambda} = \lambda/(\eta/4)$, $\hat{\gamma} = \gamma\frac{3/(4\mu)}{\sqrt{2\mu}\,\eta/4}$. Then (31) becomes $\hat{\delta} \in [0, 1]$, and $\hat{\lambda} < -\sqrt{1 - \delta^2}$. We then get, using $\nu \approx 2\sqrt{2\mu}$,

$$\mathscr{B}(z) = \frac{\sqrt{2\mu}\eta}{4}\left\{\left(-\hat{\gamma}(z_1^2 + z_2^2) - \hat{\lambda}\right)\begin{pmatrix} z_2 \\ -z_1 \end{pmatrix} - \hat{\delta}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} z_2 \\ z_1 \end{pmatrix}\right\}. \tag{32}$$

For the fixed points we have (from (10) and (11))

$$\sqrt{(z_1^2 + z_2^2)} = \frac{1}{\sqrt{-\hat{\gamma}}}\sqrt{-\hat{\lambda} \pm \sqrt{1 - \hat{\delta}^2}} \qquad =: R_\pm, \tag{33}$$

$$\frac{2z_1 z_2}{z_1^2 + z_2^2} = \hat{\delta}. \tag{34}$$

The above two equations suggest that as $(-\hat{\lambda})$ is increased, the size of the domain of attraction of z^0 increases. This could be explained as follows. When $\hat{\delta}$ is fixed, from (33) we can see that the angle the 'line joining the fixed points to the origin' makes with z_1 axis is fixed. From (34) we can see that as $(-\hat{\lambda})$ increases $R_\pm$ increases. Increases R results in increased size of the domain of attraction of z^0.

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[3] It is also confirmed by numerical simulations

Intuitively we can deduce one trend: when the damping $\hat{\delta}$ is increased $V_{0,-}$ must increase because larger damping makes it difficult for the system to reach large values. Intuition deserts us to predict other dependences, hence, we resort to numerical simulation. We fix $\mu = 1$ and $\eta = 2$. The table below contains the approximate $(V_{0,-}, V_{+,-})$ pairs obtained by the numerical procedure outlined in the previous sections: vertical axis is $\hat{\delta}$ and horizontal axis is $-\hat{\lambda}$.

$\hat{\delta}$, $-\hat{\lambda}$	1.08	1.2	1.8	2.5	5	10
0.2	(0.001, 0.17)	(0.004, 0.17)	(0.04, 0.19)	(0.11, 0.21)	(0.42, 0.23)	(1.18, 0.22)
0.4	(0.005, 0.29)	(0.014, 0.29)	(0.24, 0.31)	(0.1, 0.29)	(0.88, 0.33)	(2.5, 0.3)
0.6	(0.022, 0.29)	(0.04, 0.29)	(0.18, 0.28)	(0.4, 0.28)	(1.42, 0.28)	(3.8, 0.3)
0.8	(0.06, 0.14)	(0.09, 0.14)	(0.33, 0.14)	(0.63, 0.14)	(2.1, 0.14)	(5.2, 0.14)

Table 1. A table of approximate quasipotential $(V_{0,-}, V_{+,-})$ pairs.

As expected, as the dissipation $\hat{\delta}$ increases, the exit time $V_{0,-}$ from the domain of attraction of z^0 increases , which are represented by the first entry of the columns. Whereas $V_{+,-}$ increases and then decreases. For a fixed $\hat{\delta}$, $V_{0,-}$ increases as $(-\hat{\lambda})$ increases—possibly because the distance of the saddle from the origin increases. Whereas there is not much variability for $V_{+,-}$—possibly because $-\hat{\gamma}(z_1^2 + z_2^2) - \hat{\lambda} \approx -\sqrt{1 - \hat{\delta}^2}$ near the R_+ fixed point and this is independent of $\hat{\lambda}$; so there is not much $\hat{\lambda}$ dependence in (32) when z is close to R_+. In the shaded region $V_{0,-} > V_{+,-}$, so q_t^ε of (7) spends most of the time close to solution (13). The unshaded region has $V_{+,-} > V_{0,-}$ and so q_t^ε of (7) spends most of the time close to the solutions (14)-(15).

Conclusions

The capture of an oscillatory nonlinear system into resonance by periodic perturbations is an important process in many applications. For a fixed strength of periodic excitations and damping, weak noise makes the escape from resonance zone possible, as shown in [4]. Once outside the resonance zone damping results in decrease of action I with time. As the action decreases the system enters a different resonance zone — from results of [10] it is well known that the measure of the set of initial conditions which get trapped in the resonance zone is small. Those that get trapped, escape at a rate governed by the large-deviations principle obtained in [4] and the system evolves until it reaches close to $(q_1, q_2) = (\pm\sqrt{\mu/\gamma}, 0)$, i.e. bottom of the wells in the potential U. At the bottom of the well $I = I_b$ and $\Omega(I_b) = \sqrt{2\mu}$. In this paper we studied the dynamics of the perturbed system (2) near the bottom of the potential wells, when the forcing frequency ν is close to $2\sqrt{2\mu}$, i.e., we assume $\nu = 2\sqrt{2\mu}(1 + \varepsilon\lambda)$, where λ is a detuning parameter. The phase space is divided into domain of attractions of the solutions (13)–(15). Weak noise induces transitions between the domains of attraction. This paper developed asymptotic techniques (by combining averaging and large deviations theory) to obtain the rate functions which govern the transitions.

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